A purity theorem for the Witt group

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1. Introduction

We briefly review the definitions of quadratic spaces and Witt groups. A very detailed exposition of these topics may be found in [**8**] and in [**9**].

Let X be a scheme such that 2 is invertible in $\Gamma(\mathcal{O}_X)$. A *quadratic space* over X is a pair $\mathbf{q} = (\mathcal{E}, q)$ consisting of a locally free coherent sheaf (we also say "vector bundle") $\mathcal{E}$ and a symmetric isomorphism $q : \mathcal{E} \to \mathcal{E}^* = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$: this means that, after identifying $\mathcal{E}$ with $\mathcal{E}^{**}$ in the usual way, it satisfies $q = q^*$.

An *isometry* $\varphi : \mathbf{q} \to \mathbf{q}'$ is an isomorphism $\varphi : \mathcal{E} \to \mathcal{E}'$ such that the square

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\varphi"] \arrow[d, "q"'] & \mathcal{E}' \arrow[d, "q'"] \\
\mathcal{E}^* & \mathcal{E}'^* \arrow[l, "\varphi^*"]
\end{tikzcd}$$

commutes.

The *orthogonal sum* of $\mathbf{q}$ and $\mathbf{q}'$ is the space $\mathbf{q} \perp \mathbf{q}' = (\mathcal{E} \oplus \mathcal{E}', q \oplus q')$.

Let $\mathbf{q} = (\mathcal{E}, q)$ be a quadratic space over X and $\mathcal{F}$ a subsheaf of $\mathcal{E}$. The *orthogonal* $\mathcal{F}^\perp$ of $\mathcal{F}$ is the kernel of $q \circ i^*$, where i denotes the inclusion of $\mathcal{F}$ into $\mathcal{E}$.

A subbundle $\mathcal{L}$ of $\mathcal{E}$ is a *sublagrangian of* $\mathbf{q}$ if $\mathcal{L} \subseteq \mathcal{L}^\perp$, and it is a *lagrangian* if $\mathcal{L} = \mathcal{L}^\perp$. Note that lagrangians and sublagrangians are subbundles, i.e. locally direct factors, not just subsheaves. A space $\mathbf{q} = (\mathcal{E}, q)$ is said to be *metabolic* if it has a lagrangian.

Let $\mathrm{GW}(X)$ denote the Grothendieck group of quadratic spaces over X with respect to the orthogonal sum. Let M be the subgroup of $\mathrm{GW}(X)$ generated by metabolic spaces. The *Witt group of* X is the quotient $\mathrm{W}(X) = \mathrm{GW}(X)/M$. If $f : X \to Y$ is a map of schemes and $\mathbf{q} = (\mathcal{E}, q)$ is space over Y, the pair $f^*\mathbf{q} = (f^*\mathcal{E}, f^*q)$ is a quadratic space over X. It is easily seen that f^* respects orthogonal sums and maps metabolic spaces to metabolic spaces, thus f induces a group homomorphism $\mathrm{W}(f) : \mathrm{W}(Y) \to \mathrm{W}(X)$ and W turns out to be a contravariant functor from the category of schemes to the category of abelian groups.

If $X = \mathrm{Spec}(A)$ is affine, a quadratic space over X is the same as a pair (P, q) consisting of a finitely projective A-module P and an A-linear isomorphism $q : P \to P^*$ such that $q = q^*$. In this case a space (P, q) is metabolic if and only if it is isometric to a space of the form $\big(L \oplus L^*, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big)$.

For an affine scheme $X = \mathrm{Spec}(A)$ we denote $\mathrm{W}(X)$ by $\mathrm{W}(A)$.

Let now X be an integral scheme and $K = k(X)$ its field of rational functions. By the functoriality of W there is a canonical map $\mathrm{W}(X) \to \mathrm{W}(K)$ and, for every point $x \in X$, a canonical map $\mathrm{W}(\mathcal{O}_{X,x}) \to \mathrm{W}(K)$. We say that an element $\xi \in \mathrm{W}(K)$ is *unramified* *at* x if ξ is in the image of $\mathrm{W}(\mathcal{O}_{X,x})$. We say that an element $\xi \in \mathrm{W}(K)$ is *unramified (over* X*)* if it is unramified at every height one point $x \in X$. We say that *purity holds for* X if every unramified element of $\mathrm{W}(K)$ belongs to the image of $\mathrm{W}(X)$ in $\mathrm{W}(K)$.

Purity is known to hold for every regular integral noetherian scheme of dimension at most two [**3**] and for every regular integral noetherian affine scheme of dimension three [**14**].

The main result of this paper is the following purity theorem (§7).

Theorem A. *Purity holds for any regular local ring containing a field of characteristic* $\neq 2$.

Theorem A will be deduced from the same statement for essentially smooth local algebras over a field, using a well-known result of Dorin Popescu.

Further, using essentially the same methods, we prove (§8)

Theorem B. *Let* A *be a regular local ring containing a field of characteristic* $\neq 2$ *and* K *the field of fractions of* A. *Let* f *be a regular parameter of* A. *The natural homomorphism* $\mathrm{W}(A_f) \to \mathrm{W}(K)$ *is injective.*

From this, using a result of Piotr Jaworski for 2-dimensional regular rings, we deduce (§9)

Theorem C. *Let* A *be a regular local ring containing a field of characteristic* $\neq 2$ *and* f *a regular parameter of* A. *There is a short exact sequence*

$$0 \longrightarrow \mathrm{W}(A) \longrightarrow \mathrm{W}(A_f) \xrightarrow{\ \delta\ } \mathrm{W}(A/Af) \longrightarrow 0\ ,$$

where δ *is the restriction to* $\mathrm{W}(A_f)$ *of the second residue homomorphism* ∂_f *at the height one prime* $\mathfrak{p} = Af$.

Let $A((t)) = A[[t]]_t$ be the ring of formal Laurent series over A. As a special case of Theorem C we can formulate (§9)

Theorem D. *Let* A *be a regular local ring containing a field of characteristic* $\neq 2$. *There exists a split short exact sequence*

$$0 \to \mathrm{W}(A) \to \mathrm{W}(A((t))) \to \mathrm{W}(A) \to 0\ .$$

Remark. The method used for proving purity for an essentially smooth local k-algebra A also yields a new proof of the injectivity of $\mathrm{W}(A)$ into the Witt group $\mathrm{W}(K)$ of its field of fractions. Since this result is well-known and not very difficult (see for instance [**13**]) we use it whenever it is convenient, without proving it again.

Our proof has been inspired by Vladimir Voevodsky's work [**19**] and makes essential use of a non-degenerate trace form for finite extensions of smooth algebras, which was discovered by Leonhard Euler in a special case. We recall its definition and main properties in §§ 2 and 3.

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2. The Euler trace

Let k be any field and $A \hookrightarrow B$ a finite extension of smooth, purely d-dimensional k-algebras. Let Ω_A and Ω_B be the modules of Kähler differentials of A and B over k and let $\Omega_{B/A}$ be the module of relative differentials of B over A. Let $\omega_A = \bigwedge^d \Omega_A$, $\omega_B = \bigwedge^d \Omega_B$.

Proposition 2.1. *There exists an isomorphism of B-modules $\omega_B \xrightarrow{\simeq} \mathrm{Hom}_A(B, \omega_A)$.*

Proof. Let R be the polynomial algebra $A[X_1, \ldots, X_n]$ and $\rho : R \to B$ a surjective homomorphism of A-algebras. Let $I = \ker(\rho)$. Since B is a local complete intersection over A, by Lemma 4.4 of [**17**] there exists an isomorphism of B-modules

$$(*) \qquad \mathrm{Hom}_A(B, A) \simeq \bigwedge^n \left(\mathrm{Hom}_B(I/I^2, B)\right) .$$

On the other hand, from the canonical exact sequence of projective B-modules (see [**1**], VII, Theorem 5.8)

$$0 \to I/I^2 \to B \otimes_R \Omega_R \to \Omega_B \to 0 ,$$

we deduce, taking maximal exterior powers, that

$$(\dagger) \qquad \omega_B \otimes_B \bigwedge^n \left(I/I^2\right) \simeq B \otimes_A \omega_A .$$

From $(\dagger)$ we get, using the fact that I/I^2 is a finitely generated projective B-module,

$$\omega_B \simeq (B \otimes_A \omega_A) \otimes_B \mathrm{Hom}_B \left(\bigwedge^n (I/I^2), B \right) \simeq (B \otimes_A \omega_A) \otimes_B \bigwedge^n \left(\mathrm{Hom}_B(I/I^2, B)\right)$$

and then, from $(*)$,

$$(B \otimes_A \omega_A) \otimes_B \bigwedge^n \left(\mathrm{Hom}_B(I/I^2, B)\right) \simeq \omega_A \otimes_A \mathrm{Hom}_A(B, A) \simeq \mathrm{Hom}_A(B, \omega_A) .$$

Corollary 2.2. *If ω_A and ω_B are trivial there exists an isomorphism of B-modules*

$$\lambda : B \xrightarrow{\simeq} \mathrm{Hom}_A(B, A) .$$

The isomorphism λ induces an A-linear map

$$\mathfrak{e} : B \to A$$

defined by $\mathfrak{e}(x) = \lambda(1)(x)$. We call it *an Euler trace*, because Euler discovered a special case of it (see [**5**] and also [**16**], Chap. III). Conversely, from $\mathfrak{e}$ we get back λ as $\lambda(x)(y) = \mathfrak{e}(xy)$.

In the next proposition we record, without proof, a few obvious properties of $\mathfrak{e}$ and λ.

Proposition 2.3. *Let B be a finite locally free A-algebra and $\mathfrak{e} : B \to A$ an A-linear map such that the bilinear map*

$$\lambda : B \to \mathrm{Hom}_A(B, A) \quad \text{given by} \quad \lambda(x)(y) = \mathfrak{e}(xy)$$

is an isomorphism. Then, for every $A \to A'$, we have an A'-linear map

$$\mathfrak{e}' = \mathfrak{e} \otimes_A A' : B' = B \otimes_A A' \to A'$$

such that the associated $\lambda' : B' \to \mathrm{Hom}_{A'}(B', A')$ is an isomorphism of B'-modules. If $B = B_1 \times B_2$, λ decomposes as $\lambda_1 \times \lambda_2$, where $\lambda_i : B_i \to \mathrm{Hom}_A(B_i, A)$ is the map associated to $\mathfrak{e}|_{B_i}$. In particular, if $B = B_1 \times A$ the map $\lambda_2 : A \to A$ is the multiplication by a unit of A.

3. Traces and quadratic spaces

Let $A \hookrightarrow B$ be a finite flat extension of commutative rings. Let $\mathfrak{e} : B \to A$ be an A-linear map such that the associated $\lambda : B \to \mathrm{Hom}_A(B, A)$ is an isomorphism. To every quadratic space $\mathbf{q} = (P, q)$ over B we associate the bilinear form $\mathrm{Tr}^{\mathfrak{e}}(\mathbf{q}) = (P_A, \mathfrak{e} \circ q)$, where P_A denotes P considered as an A-module. This bilinear form is in fact a quadratic space and it is easy to check (see [**9**], I, §7) that Tr has the following properties:

(1) $\mathrm{Tr}^{\mathfrak{e}}(\mathbf{q} \perp \mathbf{q}') = \mathrm{Tr}^{\mathfrak{e}}(\mathbf{q}) \perp \mathrm{Tr}^{\mathfrak{e}}(\mathbf{q}')$.

(2) If $\mathbf{q}$ is hyperbolic, $\mathrm{Tr}^{\mathfrak{e}}(\mathbf{q})$ is hyperbolic.

(3) For any homomorphism of commutative rings $A \to A'$ we have

$$\mathrm{Tr}^{\mathfrak{e}'}(\mathbf{q} \otimes_A A') = \mathrm{Tr}^{\mathfrak{e}}(\mathbf{q}) \otimes_A A' ,$$

where $\mathfrak{e}' = \mathfrak{e} \otimes_A A'$.

(4) If, as at the end of §2, $B = B_1 \times B_2$ and $\mathfrak{e}_i = \mathfrak{e}|_{B_i}$,

$$\mathrm{Tr}^{\mathfrak{e}}(\mathbf{q}) = \mathrm{Tr}^{\mathfrak{e}_1}(\mathbf{q}_1) \perp \mathrm{Tr}^{\mathfrak{e}_2}(\mathbf{q}_2) ,$$

where $\mathbf{q}_i = \mathbf{q} \otimes_B B_i$.

(5) If, as in (4), $B = B_1 \times B_2$ but $B_2 = A$, then $\mathfrak{e}_2$ is the multiplication by a unit $u \in A^*$ and thus, for any quadratic space $\mathbf{q}$,

$$\mathrm{Tr}^{\mathfrak{e}_2}(\mathbf{q}_2) = u \cdot \mathbf{q}_2 .$$

If $f : A \to A'$ is a ring homomorphism and $B' = B \otimes_A A'$, clearly $B' = B_1' \times B_2'$ with $B_2' = A'$ and $\mathfrak{e}_2'$ is the multiplication by $f(u)$.

(6) Suppose that the map $f : A \to A'$ considered in (5) has a section $s : A' \to A$ and that $B \otimes_A A' = B' = B_1' \times B_2'$ with $B_2' = A'$. Then, by (5), $\mathfrak{e}_2'$ is the multiplication by a unit u' of A'. Replacing $\mathfrak{e}$ by $s(u')^{-1}\mathfrak{e}$ we get a new Euler map $\mathfrak{e} : B \to A$ for which $\mathfrak{e}_2' = \mathrm{id}_{A'}$ and for any ring homomorphism $A' \to A''$ we have $B'' = B_1'' \times B_2''$ with $B_2'' = A''$ and $\mathfrak{e}_2'' = \mathrm{id}_{A''}$. Thus, for any quadratic space $\mathbf{q}''$ over B''

$$\mathrm{Tr}^{\mathfrak{e}_2''}(\mathbf{q}_2'') = \mathbf{q}_2'' .$$

(7) The linear map $\mathfrak{e} : B \to A$ induces a homomorphism of Witt groups

$$\mathrm{Tr}^{\mathfrak{e}} : \mathrm{W}(B) \to \mathrm{W}(A) .$$

(8) If B is of the form $A[t]/(f) = A[\tau]$, where f is a monic polynomial of odd degree and τ the class of t, we can define an Euler map by

$$\mathfrak{e}(\tau^i) = \begin{cases} 0 & \text{if } i < n-1, \\ 1 & \text{if } i = n-1. \end{cases}$$

In this case a direct computation shows that the composite homomorphism

$$\mathrm{W}(A) \to \mathrm{W}(B) \to \mathrm{W}(A)$$

is the identity of $\mathrm{W}(A)$.

4. Reduction of purity to infinite base fields

Let $\mathbb{F}$ be a finite field of odd characteristic p and A a local, essentially smooth $\mathbb{F}$-algebra with maximal ideal $\mathfrak{m}$. Suppose that purity holds for essentially smooth local algebras over any infinite field k. Let K be the field of fractions of A and ξ an unramified element of $\mathrm{W}(K)$. Let p^m be the cardinality of $A/\mathfrak{m}$ and s an odd integer greater than 2 and prime to m. For any i let k_i be the field (in some fixed algebraic closure of $\mathbb{F}$) of degree s^i over $\mathbb{F}$. Let k be the union of all k_i. Since $k \otimes_{\mathbb{F}} (A/\mathfrak{m})$ is still a field, $B = k \otimes_{\mathbb{F}} A$ is a local, essentially smooth algebra over the infinite field k. Let $L = k \otimes_{\mathbb{F}} K$ be its field of fractions. The image ξ_L of ξ in $\mathrm{W}(L)$ is unramified. In fact, let $\mathfrak{q}$ be a height one prime of B and $\mathfrak{p} = A \cap \mathfrak{q}$. By assumption $\xi \in \mathrm{W}(A_{\mathfrak{p}})$ and since $A_{\mathfrak{p}} \to L$ factors through $B_{\mathfrak{q}}$ the class ξ_L is in $\mathrm{W}(B_{\mathfrak{q}})$ for every $\mathfrak{q}$. Since purity holds for B, $\xi_L \in \mathrm{W}(B)$. We can now find a finite subfield $\mathbb{F}'$ of k and, for $A' = \mathbb{F}' \otimes_{\mathbb{F}} A$ a $\xi' \in \mathrm{W}(A')$ which maps to ξ_L. Let K' be the field of fractions of A'. Further enlarging $\mathbb{F}'$ we may assume that the images of ξ and ξ' in $\mathrm{W}(K')$ coincide. Consider now the diagram

$$\begin{tikzcd}
\mathrm{W}(A) \arrow[r] \arrow[d] & \mathrm{W}(A') \arrow[r, "\mathrm{Tr}^{\mathfrak{e}}"] \arrow[d] & \mathrm{W}(A) \arrow[d, "\alpha"] \\
\mathrm{W}(K) \arrow[r] & \mathrm{W}(K') \arrow[r, "\mathrm{Tr}^{\mathfrak{e}}"] & \mathrm{W}(K)
\end{tikzcd}$$

where $\mathfrak{e}$ has been chosen as in §3 (8). Since the composition of the horizontal maps is the identity, we have $\alpha \circ \mathrm{Tr}(\xi') = \xi$ in $\mathrm{W}(K)$. Thus ξ is indeed in the image of $\mathrm{W}(A)$.

5. The geometric presentation lemma

We state and prove a lemma that will play a crucial role in the sequel. In geometrical disguise it sounds like this:

Lemma 5.1. *Let A be a local ring of a smooth variety over an infinite field k. Let $U = \mathrm{Spec}(A)$ and let u be the closed point of U. Let $p : \mathcal{X} \to U$ be an affine U-scheme, essentially smooth over k. Let f be a regular element of $k[\mathcal{X}]$ such that $k[\mathcal{X}]/(f)$ is finite over A. We denote by $\mathcal{X}_f$ the principal open set defined by $f \neq 0$. Assume that there exists a finite surjective morphism $\mathcal{X} \to U \times \mathbb{A}^1_k$ of U-schemes and that there exists a section $\Delta : U \to \mathcal{X}$ of p such that p is smooth along $\Delta(U)$.*

Then there exists a finite surjective morphism

$$\pi : \mathcal{X} \to U \times \mathbb{A}^1_k$$

of U-schemes with the following properties:

(a) $\pi^{-1}(U \times \{1\})$ is in $\mathcal{X}_f$.

(b) $\pi^{-1}(U \times \{0\}) = \Delta(U) \amalg \mathcal{D}$, where $\mathcal{D} \subset \mathcal{X}_f$.

Clearly the statement above is equivalent to the following, purely algebraic one.

Lemma 5.2. *Let A be a local essentially smooth algebra over an infinite field k, $\mathfrak{m}$ its maximal ideal and R an essentially smooth k-algebra, which is finite over the polynomial algebra $A[t]$. Suppose that $\epsilon : R \to A$ is an A-augmentation and let $I = \ker(\epsilon)$. Assume that R is smooth over A at every prime containing I. Given $f \in R$ such that R/Rf is finite over A we can find an $s \in R$ such that*

(1) R is finite over $A[s]$.
(2) $R/Rs = R/I \times R/J$ for some ideal J of R.
(3) $J + Rf = R$.
(4) $R(s-1) + Rf = R$.

Proof. Replacing t by $t - \epsilon(t)$ we may assume that $t \in I$. We denote by "bar" the reduction modulo $\mathfrak{m}$. By the assumptions made on R the quotient $\overline{R}$ is smooth over $\overline{A}$ at its maximal ideal $\overline{I}$. Choose an $\alpha \in R$ such that $\overline{\alpha}$ is a local parameter of the localization $\overline{R}_{\overline{I}}$ of $\overline{R}$ at $\overline{I}$. By the chinese remainders' theorem we may assume that $\overline{\alpha}$ does not vanish at the zeros of $\overline{f}$ different from $\overline{I}$. Without changing $\overline{\alpha}$ we may replace α by $\alpha - \epsilon(\alpha)$ and assume that $\alpha \in I$. Since R is integral over $A[t]$ there exists a relation of integral dependence

$$\alpha^n + p_1(t)\alpha^{n-1} + \ldots + p_n(t) = 0\ .$$

For any $r \in k^*$ and any N larger than the degree of each $p_i(t)$, putting $s = \alpha - rt^N$ we see that from the equation above that t is integral over $A[s]$. Hence R, which is integral over $A[t]$, is integral over $A[s]$. Clearly $s \in I$. To insure that $\overline{s}$ is also a local parameter of $\overline{R}_{\overline{I}}$ it suffices to take $n \geq 2$. By assumption R and $A[s]$ are both regular and since R is finite over $A[s]$, R is locally free over $A[s]$ (see for instance Corollary 18.17 of [**4**]) and hence R/Rs is free over A. Since $\overline{s}$ is a local parameter of $\overline{R}_{\overline{I}}$, $\overline{R}/\overline{s}\overline{R}$ is étale over $\overline{A}$ at the augmentation ideal $\overline{I}$ and so we can find a $g \notin I + \mathfrak{m}R$ such that $(R/Rs)_g$ is étale over A. By the next sublemma R/Rs splits as in (2).

Sublemma 5.3. *Let B be a commutative ring, $\gamma : B \to C$ a finite commutative B-algebra and $\lambda : C \to B$ an augmentation with augmentation ideal I. Let $h \in C$ be such that*
(a) C_h is étale over B.
(b) $\lambda(h)$ is invertible in B.
Then C splits as $C/I \times C/J$ for some ideal J of C.

Proof. Since $B \to C_h$ is étale and the composite map

$$B \xrightarrow{\ \gamma\ } C_h \xrightarrow{\ \lambda\ } B$$

is the identity of B, by Prop. 4.7 of [**1**], $C_h \to B$ is étale. But $C \to C_h$ is étale, hence $\lambda : C \to B$ is étale and in particular it induces an open morphism $\lambda^* : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$. Its image $\lambda^*(\mathrm{Spec}(B) = \mathrm{Spec}(C/I)$ is therefore open and since it is also closed, C splits as claimed.

To finish the proof of Lemma 5.2 we still have to choose $r \in k^*$ so that conditions (3) and (4) are satisfied. Since R/Rf is semilocal, there are only finitely many maximal ideals of R containing f. We denote by $\mathfrak{m}_1, \ldots, \mathfrak{m}_p$ those which, in case $f \in I + \mathfrak{m}R$, are different from $I + \mathfrak{m}R$. Recalling that α was chosen outside $\mathfrak{m}_1 \cup \ldots \cup \mathfrak{m}_p$, we have $s \notin \mathfrak{m}_1 \cup \ldots \cup \mathfrak{m}_p$ for almost any choice of $r \in k^*$. To see that condition (3) is satisfied it suffices to show that $J \not\subseteq \mathfrak{m}_i$ for $1 \leq i \leq p$ and that $J \not\subseteq \mathfrak{m}R + I$. The first assertion is clear because $s \in J \setminus \mathfrak{m}_i$ for $1 \leq i \leq p$. For the second one note that, since $R/Rs = R/I \times R/J$, we have $I + J = R$ and therefore $J \not\subseteq \mathfrak{m}R + I$. It remains to satisfy (4). Since R/Rf is semilocal there exists a $\lambda \in k$ such that $s - \lambda$ is invertible in R/Rf. Without perturbing conditions (1), (2) and (3) we may replace s by $\frac{1}{\lambda}s$ and thus satisfy (4) as well.

6. A commutative diagram for relative curves

Lemma 6.1. *With the notation and the hypotheses of Lemma 5.2, let* $U = \mathrm{Spec}(A)$ *and* $\mathcal{X} = \mathrm{Spec}(R)$*. Let* $p : \mathcal{X} \to U$ *be the structural morphism and* $\Delta : U \to \mathcal{X}$ *the morphism corresponding to the augmentation* $\epsilon : R \to A$*. Let* $\mathcal{Z} \subset \mathcal{X}$ *be a closed set of codimension at least 2, contained in vanishing locus of* f*. Suppose that* $\omega_{\mathcal{X}/k}$ *is trivial. There exists a nonzero element* $g \in A$ *such that* $\mathcal{X}_g \subseteq \mathcal{X} \setminus \mathcal{Z}$ *and for any such* g *there exists a commutative diagram*

$$\begin{tikzcd} \mathrm{W}(\mathcal{X} \setminus \mathcal{Z}) \arrow[r, "\psi"] \arrow[d, "\mathrm{W}(j)"'] & \mathrm{W}(U) \arrow[d, "\mathrm{W}(i)"] \\ \mathrm{W}(\mathcal{X}_g \setminus \mathcal{Z}_g) = \mathrm{W}(\mathcal{X}_g) \arrow[r, "\mathrm{W}(\Delta_g)"'] & \mathrm{W}(U_g) \end{tikzcd},$$

where $i : U_g \to U$ *and* $j : \mathcal{X}_g \to \mathcal{X} \setminus \mathcal{Z}$ *are the inclusions.*

Proof. By Lemma 5.2 there exists an element $s \in R$ satisfying the conditions (1) to (4). The evaluation in s defines a finite surjective morphism $\pi : \mathcal{X} \to U \times \mathbb{A}^1_k$ of U-schemes such that $\pi^{-1}(U \times \{0\}) = \Delta(U) \amalg \mathcal{D}_0$ with $\mathcal{D}_0 \subset \mathcal{X}_f$. Since $\omega_{U \times \mathbb{A}^1_k/k}$ is obviously trivial and $\omega_{\mathcal{X}/k}$ is trivial by assumption, we can use Corollary 2.2 to find an Euler trace $\mathfrak{e} : B \to A[t]$ such that the associated map $\lambda : B \to \mathrm{Hom}_{A[t]}(B, A[t])$ is an isomorphism. We can then choose a trace map $\mathrm{Tr} : \mathrm{W}(\mathcal{X}) \to \mathrm{W}(U \times \mathbb{A}^1_k)$ as in §3. Restricting Tr to $\mathrm{W}(\pi^{-1}(U \times \{0\})$ yields a homomorphism $\mathrm{W}(\pi^{-1}(U \times \{0\}) \to \mathrm{W}(U \times \{0\})$. Since the natural embedding $A \hookrightarrow A[t]$ is a section of the evaluation at $t = 0$, by (6) of §3 we may choose the Euler trace $\mathfrak{e} : B \to A[t]$ such that $\mathrm{Tr}|_{\mathrm{W}(\Delta(U))} = \mathrm{W}(\Delta)$.

Having fixed $\mathfrak{e}$ and Tr in this way, restricting $\mathfrak{e}$ to $\mathcal{D}_i$, $i = 1, 2$, we get trace maps $\mathrm{Tr}_i : \mathrm{W}(\mathcal{D}_i) \to \mathrm{W}(U)$. Let $\varphi_i : \mathcal{D}_i \to \mathcal{X} \setminus \mathcal{Z}$ be the inclusion. We put

$$\psi = \mathrm{Tr}_1 \circ \mathrm{W}(\varphi_1) - \mathrm{Tr}_0 \circ \mathrm{W}(\varphi_0) .$$

Since $\mathcal{Z}$ is of codimension ≥ 2 in $\mathcal{X}$ and $\pi : \mathcal{X} \to U \times \mathbb{A}^1_k$ is finite, the image of $\mathcal{Z}$ in U under the structural map is contained in the vanishing locus of some non zero $g \in A$. Making now the base change of $\mathfrak{e}$ by means of the inclusion $i : U_g \hookrightarrow U$ we get $\mathfrak{e}_g$ and Tr_g such that we still have $\mathrm{Tr}_g|_{\mathrm{W}(\Delta(U_g))} = \mathrm{W}(\Delta_g)$ (see (6) of §3). Further restricting $\mathfrak{e}_g$ to $\mathcal{D}_{ig}$, $i = 1, 2$, we get trace maps $\mathrm{Tr}_{ig} : \mathrm{W}(\mathcal{D}_{ig}) \to \mathrm{W}(U_g)$. Let $\varphi_{ig} : \mathcal{D}_{ig} \to \mathcal{X}_g \setminus \mathcal{Z}_g = \mathcal{X}_g$, $i = 1, 2$, be the inclusions. We put

$$\psi_g = \mathrm{Tr}_{1g} \circ \mathrm{W}(\varphi_{1g}) - \mathrm{Tr}_{0g} \circ \mathrm{W}(\varphi_{0g}) .$$

Clearly properties (3) and (4) of §3 imply the relation $\mathrm{W}(i) \circ \psi = \psi_g \circ \mathrm{W}(j)$. Thus, to complete the proof of the lemma, it suffices to check the relation $\psi_g = \mathrm{W}(\Delta_g)$. For this take any ξ in $\mathrm{W}(\mathcal{X}_g)$ and write a chain of relations

$$\mathrm{Tr}_g(\xi)|_{U_g \times \{1\}} - \mathrm{Tr}_g(\xi)|_{U_g \times \{0\}} = \mathrm{Tr}_{1g}(\xi|_{\mathcal{D}_{1g}}) - \mathrm{Tr}_{0g}(\xi|_{\mathcal{D}_{0g}}) - \mathrm{Tr}_g(\xi|_{\Delta(U_g)}) = \psi_g(\xi) - \mathrm{W}(\Delta_g)(\xi) .$$

A well-known theorem of Max Karoubi (see [**9**], VII, §4) asserts that for any affine k-scheme S the canonical homomorphism $\mathrm{W}(S) \to \mathrm{W}(S \times \mathbb{A}^1_k)$ is an isomorphism, and therefore the left hand side of the relation above is zero. This proves the relation $\psi_g = \mathrm{W}(\Delta_g)$, whence the commutativity of the diagram.

7. Purity

Theorem 7.1. *Let A be a local, essentially smooth algebra over an infinite field k and let K be its field of fractions. Every unramified element of $\mathrm{W}(K)$ belongs to $\mathrm{W}(A)$.*

Proof. Let $U = \mathrm{Spec}(A)$ and let ξ be an unramified element of $\mathrm{W}(K)$. By assumption there exist a smooth d-dimensional k-algebra $R = k[t_1, \dots, t_n]$ and a prime ideal $\mathfrak{p}$ of R such that $A = R_{\mathfrak{p}}$. We first reduce the proof to the case in which $\mathfrak{p}$ is maximal. To do this, choose a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$. Since k is infinite, by a standard general position argument we can find d algebraically independent elements $X_1, \dots, X_d$ such that R is finite over $k[X_1, \dots, X_d]$ and étale at $\mathfrak{m}$. After a linear change of coordinates we may assume that $R/\mathfrak{p}$ is finite over $B = k[X_1, \dots, X_m]$, where m is the dimension of $R/\mathfrak{p}$. Clearly R is smooth over B at $\mathfrak{m}$ and thus, for some $h \in R \setminus \mathfrak{m}$, the localization R_h is smooth over B. Let S be the set of nonzero elements of B, $k' = S^{-1}B$ the field of fractions of B and $R' = S^{-1}R_h$. The prime ideal $\mathfrak{p}' = S^{-1}\mathfrak{p}_h$ is maximal in R', the k'-algebra R' is smooth and $A = R'_{\mathfrak{p}'}$.

From now on we assume that $A = \mathcal{O}_{X,x}$ is the local ring of a closed point x of a smooth d-dimensional affine variety X over k.

Replacing X by a sufficiently small affine neighbourhood of x we may assume that $\omega_{X/k}$ is trivial. By Proposition 2.4 of [**3**] we may assume that ξ is defined on the complement of a closed set Z of codimension at least 2 in X. Let $f \neq 0$ be a regular function on X which vanishes on a closed set Y containing Z. By Quillen's trick (see [**15**], Lemma 5.12) we can find a morphism $q : X \to \mathbb{A}_k^{d-1}$ with the following properties:

(1) q is smooth at x.
(2) $q|_Y : Y \to \mathbb{A}_k^{d-1}$ is finite.
(3) q factors as

$$\begin{tikzcd} X \arrow[rr, "q_1"] \arrow[dr, "q"'] & & \mathbb{A}_k^d \arrow[dl, "pr"] \\ & \mathbb{A}_k^{d-1} & \end{tikzcd}$$

with q_1 finite and surjective.

Consider the cartesian square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "p_X"] \arrow[d, shift right, "p"'] & X \arrow[d, "q"] \\ U \arrow[u, shift right, "\Delta"'] \arrow[r, "r"'] & \mathbb{A}_k^{d-1} \end{tikzcd}$$

where $U = \mathrm{Spec}(\mathcal{O}_{X,x})$, $r = q|_U$, $\mathcal{X} = U \times_{\mathbb{A}_k^{d-1}} X$, p is the first projection and $\Delta : U \to \mathcal{X}$ the diagonal. Denote again by f the composition of f with p_X.

Since r is smooth, p_X is also smooth and since X is smooth over k, so is $\mathcal{X}$. By base change, condition (3) implies that $\mathcal{X}$ is an affine relative curve over U. Since U is local and q is smooth at x, p is smooth along $\Delta(U)$. From (3), by base change via

$r : U \to \mathbb{A}_k^{d-1}$, we get a commutative triangle

$$\begin{tikzcd} \mathcal{X} \arrow[r, "p_1"] \arrow[dr, "p"'] & U \times \mathbb{A}_k^1 \arrow[d] \\ & U \end{tikzcd}$$

with p_1 finite. Again by the same base change we see that $k[\mathcal{X}]/(f)$ is finite over A. Thus all the hypotheses of Lemma 5.1 are satisfied and we can find a U-morphism $\pi : \mathcal{X} \to U \times \mathbb{A}_k^1$ satisfying conditions (a) and (b).

We further claim that $\omega_{\mathcal{X}}$ is trivial. To see this observe that

$$\omega_{\mathcal{X}/k} \simeq p_X^*(\omega_{X/k}) \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_{\mathcal{X}/X}$$

and that $\omega_{\mathcal{X}/X} \simeq p^*\omega_{U/\mathbb{A}_k^{d-1}}$. Since U is essentially smooth over $\mathbb{A}_k^{d-1}$, $\omega_{U/\mathbb{A}_k^{d-1}}$ is locally free of rank one, hence trivial because U is local. Thus $p^*\omega_{U/\mathbb{A}_k^{d-1}}$ is trivial and, since $\omega_{X/k}$ is trivial by assumption, we conclude that $\omega_{\mathcal{X}/k}$ is trivial.

We can now apply Lemma 6.1 with $\mathcal{Z} = U \times_{\mathbb{A}_k^{d-1}} Z \subset \mathcal{X}$. We define $\eta = \psi(\mathrm{W}(p_X)(\xi))$ and claim that η is an extension of ξ to U. In fact, choosing $g \in A$ as in 6.1 and denoting by $i : U_g \to U$, $i' : U_g \to X \setminus Z$ and $j : \mathcal{X}_g \to \mathcal{X} \setminus \mathcal{Z}$ the inclusions, we have

$$\mathrm{W}(i)\eta = \mathrm{W}(i) \circ \psi \circ \mathrm{W}(p_X)\xi = \mathrm{W}(\Delta_g) \circ \mathrm{W}(j) \circ \mathrm{W}(p_X)\xi = \mathrm{W}(p_X \circ j \circ \Delta_g)\xi = \mathrm{W}(i')\xi \ .$$

This completes the proof of Theorem 7.1.

To prove Theorem A we now recall a celebrated result of Dorin Popescu (see [**10**], [**11**] and [**12**] or [**2**] or, for a self-contained proof, [**18**]).

Le k be a field and R a local k-algebra. We say that R is *geometrically regular* if $k' \otimes_k R$ is regular for any finite extension k' of k. A ring homomorphism $A \to R$ is called *geometrically regular* if it is flat and for each prime ideal $\mathfrak{q}$ of R lying over $\mathfrak{p}$, $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = k(\mathfrak{p}) \otimes_A R_{\mathfrak{q}}$ is geometrically regular over $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$.

Observe that any regular local ring containing a field k is geometrically regular over the prime field of k.

Popescu's theorem. *A homomorphism $A \to R$ of noetherian rings is geometrically regular if and only if R is a filtered direct limit of smooth A-algebras.*

Proof of Theorem A. Let R be a regular local ring containing a field. Let k be the prime field of R. By Popescu's theorem $R = \varinjlim A_\alpha$, where the A_α's are smooth k-algebras. We first observe that we may replace the direct system of the A_α's by a system of essentially smooth local k-algebras. In fact, if $\mathfrak{m}$ is the maximal ideal of R, we can replace each A_α by $(A_\alpha)_{\mathfrak{p}_\alpha}$, where $\mathfrak{p}_\alpha = \mathfrak{m} \cap A_\alpha$. Note that in this case the canonical morphisms $\varphi_\alpha : A_\alpha \to R$ are local and every A_α is a regular local ring, in particular a factorial ring.

Let now L be the field of fractions of R and, for each α let K_α be the field of fractions of A_α. Let ξ be an unramified element of $\mathrm{W}(L)$. We may represent ξ by a diagonal matrix $q = \mathrm{diag}(r_1, \ldots, r_n)$ with $r_1, \ldots, r_n$ in R. Let Σ be the (finite) set of height one primes of R which divide at least one of the r_i. For every $\mathfrak{p} \in \Sigma$ we can find a matrix

$\sigma(\mathfrak{p}) \in Gl_n(L)$ that transforms q into a diagonal form $\operatorname{diag}(u_1(\mathfrak{p}), \ldots, u_n(\mathfrak{p}))$ with every $u_i(\mathfrak{p}) \in R \setminus \mathfrak{p}$. Clearing denominators we may assume that $\sigma(\mathfrak{p}) \in M_n(R)$ and that

$$\sigma(\mathfrak{p})^T q \sigma(\mathfrak{p}) = \operatorname{diag}(u_1(\mathfrak{p}), \ldots, u_n(\mathfrak{p}))(d(\mathfrak{p}))^2$$

for some $d(\mathfrak{p}) \in R$. We can now choose an index α such that, for every $\mathfrak{p} \in \Sigma$, A_α contains preimages $\tilde{r}_1, \ldots, \tilde{r}_n$, $\tilde{u}_1(\mathfrak{p}), \ldots, \tilde{u}_n(\mathfrak{p})$, $\tilde{d}(\mathfrak{p})$ and $\tilde{\sigma}_{ij}(\mathfrak{p})$ of the elements $r_1, \ldots, r_n$, $u_1(\mathfrak{p}), \ldots, u_n(\mathfrak{p})$, $d(\mathfrak{p})$ and of the coefficients $\sigma_{ij}(\mathfrak{p})$ of $\sigma(\mathfrak{p})$. Having chosen these preimages consider the relations

$$(\star) \qquad \tilde{\sigma}(\mathfrak{p})^T \tilde{q} \tilde{\sigma}(\mathfrak{p}) = \operatorname{diag}(\tilde{u}_1(\mathfrak{p}), \ldots, \tilde{u}_n(\mathfrak{p}))(\tilde{d}(\mathfrak{p}))^2$$

where $\tilde{q} = \operatorname{diag}(\tilde{r}_1, \ldots, \tilde{r}_n)$ and $\tilde{\sigma}(\mathfrak{p})$ is the matrix $(\tilde{\sigma}_{ij}(\mathfrak{p}))$. Since they hold over R, we may assume, after replacing α by some larger index, that they hold over A_α. We claim that the class of $\tilde{q}$ (which we still denote by $\tilde{q}$) is an unramified element of $\mathrm{W}(K_\alpha)$. To show this suppose that $\tilde{q}$ is ramified at a height one prime ideal pA_α. Then p divides some $\tilde{r}_i$. Any height one prime $\mathfrak{p}$ of R containing pR also contains r_i and thus belongs to Σ. Since $u_i(\mathfrak{p}) \in R \setminus \mathfrak{p}$ we have $\tilde{u}_i(\mathfrak{p}) \in A_\alpha \setminus pA_\alpha$ and thus the relation $(\star)$ shows that $\tilde{q}$ is unramified at pA_α. By purity for A_α there exists a $\xi_\alpha \in \mathrm{W}(A_\alpha)$ that coincides with $\tilde{q}$ in $\mathrm{W}(K_\alpha)$. The ideal $\mathfrak{r} = \ker(\varphi_\alpha)$ is prime and does not contain any $\tilde{r}_i$. Hence $\tilde{q}$ is a quadratic space over the essentially smooth local algebra $B_\alpha = (A_\alpha)_\mathfrak{r}$. Since $\tilde{q}$ and ξ_α coincide in $\mathrm{W}(K_\alpha)$ they already coincide in $\mathrm{W}(B_\alpha)$ because $\mathrm{W}(B_\alpha) \to \mathrm{W}(K_\alpha)$ is injective. The commutative diagram of ring homomorphisms

$$\begin{array}{ccc} A_\alpha & \xrightarrow{\varphi_\alpha} & R \\ \downarrow & & \downarrow \\ B_\alpha & \longrightarrow & L \end{array}$$

shows that $\mathrm{W}(\varphi_\alpha)(\xi_\alpha) = q$ in $\mathrm{W}(L)$. This proves that q is indeed in $\mathrm{W}(R)$.

8. An injectivity theorem

If A is a regular ring of dimension greater than 3 and K its field of fractions, the canonical homomorphism $\mathrm{W}(A) \to \mathrm{W}(K)$ need not be injective. In this section we prove the following injectivity result, from which we shall deduce Theorem C.

Theorem 8.1. *Let A be a local, essentially smooth algebra over an infinite field k of characteristic $\neq 2$. Let K be the field of fractions of A and f a regular parameter of A. The canonical homomorphism $\mathrm{W}(A_f) \to \mathrm{W}(K)$ is injective.*

The proof of this theorem is similar to that of Theorem 7.1. As we did there, we assume, without loss of generality, that A is the local ring of a closed point x of a smooth affine variety X. If A is 1-dimensional $A_f = K$ and there is nothing to prove, so we assume that A is at least 2-dimensional. We need the following variant of Quillen's trick.

Lemma 8.2. *Let X be an irreducible affine smooth variety over an infinite field k and x a closed point of X. Let A be the local ring of x, $f \in k[X]$ a regular function on X which is a regular parameter of A and $g \in k[X]$, g prime to f. Denote by Y the vanishing* *locus of f and by Z the vanishing locus of g. There exists a morphism $q : X \to \mathbb{A}_k^{d-1}$ with the following properties:*

(1) q is smooth at x.
(2) $q|_{Y\cap Z} : Y \cap Z \to \mathbb{A}_k^{d-1}$ is finite.
(3) q factors as

$$\begin{tikzcd} X \arrow[rr, "q_1"] \arrow[dr, "q"'] & & \mathbb{A}_k^d \arrow[dl, "pr"] \\ & \mathbb{A}_k^{d-1} & \end{tikzcd}$$

with q_1 finite and surjective.
(4) $q(Y) = \{0\} \times \mathbb{A}_k^{d-2}$
(5) $q^{-1}\left(\{0\} \times \mathbb{A}_k^{d-2}\right) = Y \cup Y'$ for some closed set $Y' \subset X$ which avoids x.

We first recall an auxiliary result, which has been proved in slightly different versions by several authors.

Lemma 8.3. *Under the assumptions of Lemma 8.2 there exists a morphism $q_2 : X \to \mathbb{A}_k^d$ such that*

(i) q_2 is finite.
(ii) q_2 is étale at x.
(iii) $k(x) = k(q_2(x))$.
(iv) $Y \cap q_2^{-1}(q_2(x)) = \{x\}$.

Proof. Suppose that X is a closed set of $\mathbb{A}_k^N \subset \mathbb{P}^N$ and let $\overline{X}$ be its closure in $\mathbb{P}^N$. To prove Lemma 8.3 we will take for q_2 the projection from a suitable linear subspace L at infinity. Let $\overline{k}$ be an algebraic closure of k and $\varphi : \overline{k} \otimes_k X \to X$ the canonical projection. Then $\varphi^{-1}(x)$ is a finite set of closed points $\{x_1, \dots, x_n\}$ of $\overline{k} \otimes_k X$. Choose an $N-d-1$-dimensional linear subspace L in $\mathbb{P}^N \setminus \mathbb{A}_k^N$ with the following properties:
(a) L is defined over k.
(b) L does not intersect $\overline{k} \otimes_k \overline{X}$.
(c) L does not intersect the tangent planes of $\overline{k} \otimes_k \overline{X}$ at $x_1, \dots, x_n$.
(d) For $i \neq j$ we have $q_2(x_i) \neq q_2(x_j)$.
(e) L does not intersect the closures of the cones with vertices $x_1, \dots, x_n$ and base $\overline{k} \otimes_k Y$.

Dimension considerations show the existence of infinitely many such linear spaces. Condition (a) insures that q_2 is defined over k. Condition (b) insures that $q_2 : X \to \mathbb{A}_k^d$ is finite. Condition (c) insures that q_2 is étale at x. Since the group $\mathrm{Aut}_k(\overline{k})$ acts transitively on $\{x_1, \dots, x_n\}$, by condition (d) it acts transitively on $\{q_2(x_1), \dots, q_2(x_n)\}$ as well. This shows that the separability degree of $k(q_2(x))$ over k is the same as that of $k(x)$. But q_2 is étale at x, hence the extension $k(x)/k(q_2(x))$, being separable, must be of degree one. Thus condition (iii) is satisfied. Finally, condition (iv) follows from (e).

Proof of Lemma 8.2. We choose q_2 as in the previous lemma. We put $B = k[\mathbb{A}_k^d]$ and $C = k[X]$. The map q_2 induces an inclusion $\iota : B \hookrightarrow C$ and C is a finite B-module. The images of the closed subschemes $Y = \{f = 0\}$ and $Z = \{g = 0\}$ of X are two closed closed subschemes of $\mathbb{A}_k^d$ defined, respectively, by $f_0 = 0$ and $g_0 = 0$ for some $f_0, g_0 \in k[\mathbb{A}_k^d]$. The inclusion ι induces a finite map $B/Bf_0 \to C/Cf$. Let $\mathfrak{m}$ be the maximal ideal of B corresponding to the closed point $q_2(x)$. Since x is the unique closed point of Y lying over $q_2(x)$, the localization $(C/Cf)_{\mathfrak{m}} = B_{\mathfrak{m}} \otimes_B (C/Cf)$ is local and finite over $(B/Bf_0)_{\ m}$. By condition (iii) these two local rings have the same residue field, hence by Nakayama's lemma they are isomorphic. This shows in particular that f_0 is a regular parameter of B at $q_2(x)$. On the other hand, since C is étale over B at x, f_0 is also a regular parameter of C at x.

We now have two polynomials f_0 and g_0 in $B = k[X_1, \ldots, X_d]$ which we may assume monic in, say, X_1. The map $k[Y_1, \ldots, Y_d] \to k[X_1, \ldots, X_d]$ defined by $Y_1 \mapsto f_0$ and $Y_i \mapsto X_i$ for $i \neq 1$ induces a finite morphism $q_3 : \mathbb{A}^d_k \to \mathbb{A}^d_k$. Composing q_2 with q_3 we obtain a finite map $q_1 = q_3 \circ q_2 : X \to \mathbb{A}^d_k$. This map is smooth at x because q_2 is étale at x and f_0 is a regular parameter at $q_2(x)$. It maps Y onto the hyperplane $Y_1 = 0$ and Z onto some closed set $\{g_1 = 0\}$. Since q_1 is a local-étale isomorphism at x, Y_1 does not divide g_1. Thus $q_1(Y \cap Z)$ is a proper closed subset of the hyperplane $Y_1 = 0$. We may therefore assume, after a linear change of coordinates involving only $Y_2, \ldots, Y_d$, that the projection pr onto $Y_2 = 0$ is finite on $q_1(Y \cap Z)$. We now take $q = pr \circ q_1$.

Since $q^{-1}\left(\{0\} \times \mathbb{A}^{d-2}_k\right)$ is smooth at x, it contains only one component —namely Y— that passes through x, whence (5).

Proof of Theorem 8.1. Let ξ be an element in the kernel of $\mathrm{W}(A_f) \to \mathrm{W}(K)$. There is a $g \in A$, which we may suppose prime to f, such that $\xi \in \ker(\mathrm{W}(A_f) \to \mathrm{W}(A_{fg}))$ We may represent ξ by a quadratic space $\mathbf{q}$ defined over A_f which becomes hyperbolic over A_{fg}. Patching $\mathbf{q}$ over $\mathrm{Spec}(A_f)$ with a suitable hyperbolic space over $\mathrm{Spec}(A_g)$ we get a space over the complement of the closed set $W = Y \cap Z$, where $Y = \{f = 0\}$ and $Z = \{g = 0\}$. Applying Lemma 8.2 we get a map $q : X \to \mathbb{A}^{d-1}_k$ satisfying properties (1) to (5). Let $h \in k[X]$ be an element which vanishes identically on W and such that q is finite on the closed subscheme defined by $\{h = 0\}$. As in the proof of Theorem 7.1, but with h instead of f and W instead of Z, we get a commutative square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "p_X"] \arrow[d, shift right, "p"'] & X \arrow[d, "q"] \\ U \arrow[u, shift right, "\Delta"'] \arrow[r, "r"'] & \mathbb{A}^{d-1}_k \end{tikzcd}$$

where $U = \mathrm{Spec}(\mathcal{O}_{X,x})$, $r = q|_U$, $\mathcal{X} = U \times_{\mathbb{A}^{d-1}_k} X$, p is the first projection and $\Delta : U \to \mathcal{X}$ the diagonal. We denote again by h the composition of h with p_X and we put $\mathcal{W} = U \times_{\mathbb{A}^{d-1}_k} W$. As in the proof of 7.1, we assume that X has been so chosen that ω_X is trivial.

Applying the geometric presentation lemma we find a map $\pi : \mathcal{X} \to U \times \mathbb{A}^1_k$ of U-schemes such that $\pi^{-1}(U \times \{1\}) = \mathcal{D}_1$ is in $\mathcal{X}_h$ and $\pi^{-1}(U \times \{0\}) = \Delta(U) \amalg \mathcal{D}_0$, where $\mathcal{D}_0 \subset \mathcal{X}_h$. Put for simplicity $s = Y_2$. By condition (5) we have $\mathcal{W} \subset \mathcal{X} \setminus \mathcal{X}_s$ and hence, by Lemma 6.1, there exists a commutative square

$$\begin{tikzcd} \mathrm{W}(\mathcal{X} \setminus \mathcal{W}) \arrow[r, "\psi"] \arrow[d, "\mathrm{W}(j)"'] & \mathrm{W}(U) \arrow[d, "\mathrm{W}(i)"] \\ \mathrm{W}(\mathcal{X}_s) \arrow[r, "\mathrm{W}(\Delta_s)"'] & \mathrm{W}(U_s) \end{tikzcd},$$

where $i : U_s \to U$ and $j : \mathcal{X}_s \to \mathcal{X} \setminus \mathcal{W}$ are the inclusions. Repeating the argument of the proof of Theorem 7.1, we define $\eta = \psi(\mathrm{W}(p_X)(\xi)) \in \mathrm{W}(A)$ and get $\eta_s = \xi_s$. By condition (5), $A_s = A_f$ and since $\mathrm{W}(A) \to \mathrm{W}(K)$ is injective and ξ vanishes on $\mathrm{W}(K)$ we get $\eta = 0$. This shows that $\xi = 0$ as well.

Proof of Theorem B. We first extend Theorem 8.1 to the case of an infinite base field. This is even simpler than for Theorem A: we find a sufficiently large odd degree extension $\mathbb{F}'$ of the finite base field $\mathbb{F}$ such that $A' = \mathbb{F}' \otimes_{\mathbb{F}} A$ is still a local ring and $\xi_{\mathbb{F}'} = 0$ in $\mathrm{W}(A')$. Then, choosing $\mathfrak{e}$ as in §3, (8), we see that $\xi = \mathrm{Tr}^{\mathfrak{e}}(\xi_{\mathbb{F}'}) = 0$.

We now prove Theorem B. Let R be a regular local ring containing a field and let L be the field of fractions of R. Let k be the prime field of R. As in the proof of Theorem A, $R = \varinjlim A_\alpha$, where the A_α's are essentially smooth local k-algebras. Let f be a regular parameter of R and ξ an element in the kernel of $\mathrm{W}(R_f) \to \mathrm{W}(L)$. There exists a $g \in R$ such that ξ vanishes in $\mathrm{W}(R_{fg})$. For a suitable index α choose lifts f_α and g_α of f and g in A_α. We may replace the filtered direct system of the A_α by the subsystem of all A_β with $\beta \geq \alpha$. Clearly we still have $R = \lim\limits_{\beta \geq \alpha} A_\beta$. We put, for every $\beta \geq \alpha$, $f_\beta = \varphi_{\beta\alpha}(f_\alpha)$ and $g_\beta = \varphi_{\beta\alpha}(g_\alpha)$ where the $\varphi_{\beta\alpha} : A_\alpha \to A_\beta$ are the transition homomorphisms. It is easy to see that $\lim\limits_{\beta \geq \alpha} (A_\beta)_{f_\beta} = R_f$ and $\lim\limits_{\beta \geq \alpha} (A_\beta)_{f_\beta g_\beta} = R_{fg}$. Since the functor W commutes with filtered direct limits, we have

$$\lim_{\beta \geq \alpha} \ker\left(\mathrm{W}\left((A_\beta)_{f_\beta}\right) \to \mathrm{W}\left((A_\beta)_{f_\beta g_\beta}\right)\right) = \ker\left(\mathrm{W}(R_f) \to \mathcal{W}(R_{fg})\right).$$

Since $\varphi_\beta : A_\beta \to R$ is local, f_β is a regular parameter of A_β. Hence the left hand side vanishes and, in particular, $\xi = 0$. This proves Theorem B.

9. A short exact sequence

Let B be a discrete valuation ring, $\mathfrak{p} = Bp$ its maximal ideal and L its field of fractions. Let $v : L^* \to \mathbb{Z}$ be the corresponding valuation of L. Recall that there is a homomorphism (which depends on the choice of the local parameter p) $\partial_p : \mathrm{W}(L) \to \mathrm{W}(B/\mathfrak{p})$ called *second residue* and defined on rank one forms $< up^m >$ with $u \in B^*$ by

$$\partial_p(< up^m >) = \begin{cases} 0 & \text{if } m \text{ is even ,} \\ < \overline{u} > & \text{if } m \text{ is odd,} \end{cases}$$

where $\overline{u}$ is the image of u in $B/\mathfrak{p}$.

This homomorphism fits into the exact sequence

$$0 \longrightarrow \mathrm{W}(B) \longrightarrow \mathrm{W}(L) \xrightarrow{\partial_p} \mathrm{W}(B/\mathfrak{p}) \longrightarrow 0\,.$$

Proof of Theorem C. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{W}(A) \arrow[r, "\epsilon"] \arrow[d, "\alpha"'] & \mathrm{W}(A_f) \arrow[r, dotted, "\delta"] \arrow[d, "\beta"] & \mathrm{W}(A/Af) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathrm{W}(A_{\mathfrak{p}}) \arrow[r, "\iota"] & \mathrm{W}(K) \arrow[r, "\partial_f"] & \mathrm{W}(k(\mathfrak{p})) \arrow[r] & 0
\end{tikzcd}$$

of solid arrows in which the bottom line is exact. We first want to show that

$$\partial_f \circ \beta\left(\mathrm{W}(A_f)\right) \subseteq \mathrm{W}(A/Af)$$

and then check that the top line is exact.

For the first assertion it suffices to show, by purity, that, for any $\xi \in \mathrm{W}(A_f)$, $\partial_f \circ \beta(\xi)$ is unramified over A/Af. Let $\mathfrak{q}/Af$ be a prime of height one of A/Af. We want to show that $\partial_f \circ \beta(\xi)$ is in the image of $\mathrm{W}(A_\mathfrak{q}/A_\mathfrak{q}f)$. For this, after replacing A by $A_\mathfrak{q}$ in the diagram above, we may assume that A is a local regular ring of dimension 2. But in this case the assertion is precisely Theorem 3 of [**7**].

Exactness left and right is obvious. Let ξ be an element of $\ker(\delta)$. Since β is injective, we may consider ξ as an element of $\mathrm{W}(K)$. From the exactness of the bottom line we see that ξ is in the image of $\mathrm{W}(A_\mathfrak{p})$. Since it also belongs to $\beta(\mathrm{W}(A_f))$, it is unramified and by purity it comes from $\mathrm{W}(A)$.

Proof of Theorem D. Apply Theorem C to the local ring $A[[t]]$, taking t as regular parameter and using the fact that $\mathrm{W}(A[[t]]) = \mathrm{W}(A)$.

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